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Phase Transition for a One-Dimensional Lattice Gas with Hard Core

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Existence of a phase transition is proved for a one-dimensional lattice gas with long-range interaction and nearest neighbor exclusion.

KEY WORDS: One-dimensional; phase transition; lattice gas; hard-core; long-range interaction.

1. INTRODUCTION

Using several versions of Peierls' method,[(4)] Dobrushin[(2)] established the existence of phase transitions for various lattice gas models of dimension $d \geqslant 2$. Included in ref. 2 was a proof of the existence of a phase transition for a lattice gas with hard core. Subsequently, Fröhlich and Spencer,[(3)] using a substantial modification of this basic contour method, proved the existence of a spontaneous magnetization at low temperature for the one dimensional Ising model with $1/r^2$ interaction energy (but no hard core).

In this paper we combine the techniques of ref. 2 and 3 to give a proof of the existence of a phase transition for a one-dimensional pair potential with hard core whose interaction strength decays like $1/r^2$. The imposition of the hard-core condition on the Hamiltonian has the effect of making the Hamiltonian less symmetric with respect to the values of the occupation variables $\{x_i\}$ (see (1.1) below).

We note that significant advances in the study of one-dimensional long-range ferromagnetic lattice spin models have been made since the publication of ref. 3. Applications of percolation theory, using the

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Fortuin–Kesteleyn representation, to one-dimensional Ising and Potts models were recently given in ref. 7, which also contains a useful summary and listing of other related work on one-dimensional models.

Let $\mathscr{C}$ denote the set of "allowable" configurations of the form $x = \{x_i\}_{i \in \mathbf{Z}}$ with $x_i = 0$ or 1 such that $x_i x_{i+1} = 0$ for all $i \in \mathbf{Z}$. The Hamiltonian for volume $[-L, L]$ is given by

$$\frac{1}{4} H(x) = \frac{1}{4} H_L(x) = -\sum_{-L \leqslant i < j \leqslant L} \frac{(-1)^{i-j}}{|i-j|^2} x_i x_j - \sum_{|i| > L} \sum_{|j| \leqslant L} \frac{(-1)^{i-j}}{|i-j|^2} \times x_i x_j - \mu \sum_{i=-L}^{L} x_i \tag{1.1}$$

for $x \in \mathscr{C}$, where for $|i| > L$ we will assume the boundary condition

$$x_i = \begin{cases} 1 & \text{if} \quad i \text{ is even} \\ 0 & \text{if} \quad i \text{ is odd} \end{cases} \tag{1.2}$$

or alternatively

$$x_i = \begin{cases} 1 & \text{if} \quad i \text{ is odd} \\ 0 & \text{if} \quad i \text{ is even} \end{cases} \tag{1.3}$$

If $x \notin \mathscr{C}$, we may assume $H(x) = \infty$ with the convention $\exp[-\beta \cdot \infty] = 0$ when $\beta > 0$. The parameter μ in (1.1) represents the chemical potential for this lattice gas model. For fixed μ and inverse temperature β, denote by $\langle \cdot \rangle_L^+ (\beta, \mu)$ [respectively $\langle \cdot \rangle_2^- (\beta, \mu)$] the finite-volume equilibrium states for $\beta H_L(x)$ with boundary condition (1.2) [respectively (1.3)]. We can now state the main result.

Theorem 1.1. Let $\alpha = \sum_{j \geqslant 2} (-1)^{j+1}/j^2$. If $\mu > \alpha$, then for all β sufficiently large

$$\langle x_0 \rangle_L^+ (\beta, \mu) > 1/2 \tag{1.4}$$

and

$$\langle x_0 \rangle_L^- (\beta, \mu) < 1/2 \tag{1.5}$$

uniformly in L.

From Theorem 1.1 it follows by the methods of Ruelle,[5] for example, that two distinct extremal Gibbs states exist for H, β, μ when $\mu > \alpha$ and when β is sufficiently large. From Dobrushin's uniqueness theorem[1] (see also ref. 6) it follows that the Gibbs state is unique if

$$\beta \sum_{j \geqslant 2} j^{-2} < 2$$

Hence the system experiences a phase transition.

The proof of Theorem 1.1 is given in Section 3, where reference is made to some results of Fröhlich and Spencer.[3] We list these results in Section 2 for the convenience of the reader.

2. REVIEW OF FRÖHLICH AND SPENCER[3]

In this section we list some definitions and theorems of Fröhlich and Spencer[3] which will be used in the proof of Theorem 1.1.

Let

$$\tilde{H}_L(\sigma) = \sum_{i<j} J_{ij}(1 - \sigma_i \sigma_j) \tag{2.1}$$

where $\sigma_i = \pm 1$ for $i \in [-L, L]$, $\sigma_i = +1$ for $|i| > L$, and for fixed $c > 0$,

$$J_{ij} = \begin{cases} c & \text{if } \; |i-j| = 1 \\ |i-j|^{-2} & \text{if } \; |i-j| \geqslant 2 \end{cases} \tag{2.2}$$

We note that Fröhlich and Spencer considered the case $c = 1$, but the theorems below are valid for (2.2) with no significant changes in the proofs.

Let $\mathbf{Z}^*$ be the lattice of nearest neighbor bonds, $b = (i, i+1)$ for $i \in \mathbf{Z}$. Each configuration σ of spins uniquenly specifies a contour (or collection of spin flips) $\Gamma = \Gamma(\sigma) \subset \mathbf{Z}_L^* = \mathbf{Z}^* \cap [-L-1, L+1]$ with

$$b \in \Gamma \qquad \text{iff} \quad \sigma_i \sigma_{i+1} = -1 \tag{2.3}$$

The collection of such contours with even cardinality is in one-to-one correspondence with possible configurations. Thus, we may make the identifications

$$\tilde{H}_L(\sigma) = \tilde{H}_L(\Gamma) \equiv \tilde{H}_L(\Gamma(\sigma)) \tag{2.4}$$

Definition 2.1. Let $\gamma = \{(i_1, i_1 + 1), \ldots, (i_n, i_n + 1)\}$ be an arbitrary collection of spin flups with $i_k < i_{k+1}$ for $k = 1, \ldots, n-1$.

(a) Let $d(\gamma) = i_n + 1 - i_1$.

(b) Let $I(\gamma)$ be the open interval $(i_1, i_n + 1)$.

(c) Let I_k be the open interval $(i_k, i_{k+1}+1)$.

(d) Let $\bar{I}_k$ be the closed interval $[i_k, i_{k+1}+1]$.

(e) Let

$$L(\gamma) = \sum_{k=1}^{n} \{[\ln_2(i_{k+1} - i_k)] + 1\}$$

where $[\ln_2(i_{k+1} - i_k)]$ is the integer part of the logarithm base 2 of $i_{k+1} - i_k$.

(f) The distance $\text{dist}[(i_k, i_k+1), (i_m, i_m+1)]$ between the spin flups (i_k, i_k+1) and (i_m, i_m+1) is $[i_m - i_k]$, the distance in $\mathbf{R}$ between the midpoints of the spin flips.

For any $M > 0$, any even collection of spin flips Γ can be partitioned into disjoint subsets $\gamma_1, \gamma_2,\ldots$ called primitive contours, which satisfy the following condition.

Condition D:

(a) The cardinality of each γ_α is even, $\bigcup_{\alpha \geq 1} \gamma_\alpha = \Gamma$, and $\gamma_\alpha \cap \gamma_{\alpha'} = \varnothing$ for $\alpha \neq \alpha'$.

(b) $\text{dist}(\gamma_\alpha, \gamma_{\alpha'}) \geq M[\min(d(\gamma_\alpha), d(\gamma_{\alpha'}))]^{3/2}$ for $\alpha \neq \alpha'$.

(c) If $\gamma \subset \gamma_\alpha$ and $\text{dist}(\gamma, \gamma_\alpha \backslash \gamma) \geq 2M\, d(\gamma)^{3/2}$, then card (γ) is odd for all α.

In Condition D, M is independent of Γ and each γ_α, and will be chosen later.

The following results, among others, were proved by Fröhlich and Spencer[3] and were used by them to prove the existence of a phase transition for the Hamiltonian (2.1).

From Theorem B, Lemma 2.1, and the remark in Section 3 of ref. 3, one obtains the following.

Theorem 2.1. If γ_α satisfies Condition D(c) and card (γ_α) is even, then

$$\tilde{H}_L(\gamma_\alpha) \geq \frac{c_1}{(\ln M)^2} L(\gamma_\alpha)$$

where $c_1 > 0$ and independent of M and γ_α.

The proof of Theorem C in ref. 3 establishes the following.

Theorem 2.2. There exists a constant c_2 independent of R and L such that

$$\text{card}\{\gamma \in \mathbf{Z}_L^*\colon L(\gamma) \leq R,\ I(\gamma) \ni 0\} \leq c^{c_2 R}$$

Theorem 2.3 below is Theorem 4.1 of ref. 3.

Theorem 2.3. Let $\Gamma = \gamma \cup \gamma_2 \cup \gamma_3 \cup \cdots$ satisfy Condition D. Then there is a constant c_3 independent of M such that

$$\tilde{H}(\Gamma \backslash \gamma) + \tilde{H}(\gamma) - \tilde{H}(\Gamma) \leqslant c_3 \frac{\ln M}{M} L(\gamma)$$

The following corollary comes from the proof of Theorem 4.1 in ref. 3.

Corollary 2.4. With the same hypotheses as in Theorem 2.3, let I_k be defined for γ as in Definition 2.1c and let

$$A_k = \{(i, j) \mid i \in I(\gamma_\alpha) \text{ for some } \gamma_\alpha \text{ such that } I(\gamma_\alpha) \subset I_k \text{ and } j \notin I_k\}$$

Define

$$\chi_{A_k}(i, j) = \begin{cases} 1 & \text{if} \quad (i, j) \in A_k \\ 0 & \text{if} \quad (i, j) \notin A_k \end{cases}$$

Then

$$2 \sum_{i<j} J_{ij} [\chi_{A_k}(i, j) + \chi_{A_k}(j, i)] \leqslant \frac{c_4}{M} \ln(i_{k+1} - i_k)$$

for some constant $c_4 > 0$.

3. PROOF OF THEOREM 1.1

Lemma 3.1. For an allowable configuration $x \in \mathscr{C}$, the Hamiltonian $H(x)$ given by (1.1) for volume $V = [-L, L]$ with boundary condition (1.2) is equal, to within an additive constant, to

$$H_L(\sigma) = \sum_{i<j} J_{ij}(1 - \sigma_i \sigma_j) \tag{3.1}$$

under the transformation

$$\sigma_i = (-1)^i (2x_i - 1) \tag{3.2}$$

The boundary condition (1.2) becomes $\sigma_i \equiv +1$ for $|i| > L$ and

$$J_{ij} = \begin{cases} \mu - \alpha & \text{if} \quad |i-j| = 1 \\ |i-j|^{-2} & \text{if} \quad |i-j| \geqslant 2 \end{cases} \tag{3.3}$$

with α given in Theorem 1.1.

Proof. Let $|\tilde{\Gamma}(x)|$ be the number of pairs (x_i, x_{i+1}) in x such that $x_i = x_{i+1}$ and let $|\Gamma(x)|$ be the number of pairs (x_i, x_{i+1}) such that $x_i \neq x_{i+1}$, i, $i+1 \in [-L-1, L+1]$. Then with the boundary condition (1.2),

$$2 \sum_{i=-L}^{L} x_i + 1 - (-1)^L = |\Gamma(x)| \tag{3.4}$$

and $|\Gamma(x)| + |\tilde{\Gamma}(x)| = 2L+2$. Thus,

$$\sum_{i=-L}^{L} x_i = -\frac{1}{2} |\tilde{\Gamma}(x)| + L + \frac{1+(-1)^L}{2}$$

Equation (1.1) then becomes

$$H(x) = -4 \sum_{\substack{(i,j) \in U_L \\ |i-j|>1}} \frac{(-1)^{i+j}}{|i-j|^2} x_i x_j + 2\mu |\tilde{\Gamma}(x)| + c_1 \tag{3.5}$$

where c_1 is a constant and where

$$U_L = \{(i, j) \mid i < j,\ i \text{ or } j \text{ is in } [-L, L]\}$$

Substituting (3.2) into (3.5) gives

$$H_L(x) = H_L(\sigma) = - \sum_{\substack{(i,j) \in U_L \\ |i-j|>1}} \sigma_i \sigma_j |i-j|^{-2} + 2(\mu - \alpha) |\Gamma(\sigma)| + c_2 \tag{3.6}$$

where c_2 is a constant and $|\Gamma(\sigma)| = |\tilde{\Gamma}(x)|$ is the number of pairs (σ_i, σ_{i+1}) in σ such that $\sigma_i \neq \sigma_{i+1}$ for i, $i+1 \in [-L-1, L+1]$. Thus, we may write

$$\sum_{\substack{i<j \\ |i-j|=1}} (\sigma_i - \sigma_j)^2 = 4 |\Gamma(\sigma)|$$

or

$$|\Gamma(\sigma)| = -\frac{1}{2} \sum_{\substack{(i,j) \in U_L \\ |i-j|=1}} \sigma_i \sigma_j + c_3 \tag{3.7}$$

where c_3 depends only on L. Substituting (3.7) into (3.6) gives

$$H(\sigma) = - \sum_{\substack{(i,j) \in U_L \\ |i-j|>1}} \sigma_i \sigma_j (i-j)^{-2} - (\mu - \alpha)$$

$$\times \sum_{\substack{(i,j) \in U_L \\ |i-j|=1}} \sigma_i \sigma_j + 2(\mu - \alpha) c_3 + c_2 \tag{3.8}$$

Rewriting (3.8) gives us

$$H(\sigma)=\sum_{i<j} J_{ij}(1-\sigma_i\sigma_j)-c_4$$

where $c_4=c_2+2(\mu-\alpha)c_3-\sum_{(i,j)\in U_L} J_{ij}$.

Remark 3.1. If the boundary condition (1.2) is replaced by (1.3), the proof of Lemma 3.1 is modified by replacing (3.4) by

$$2\sum_{i=-L}^{L} x_i=|\Gamma(x)|-1-(-1)^L$$

Remark 3.2. Our use of $|\Gamma(x)|$ and $|\tilde{\Gamma}(x)|$ in the proof of Lemma 3.1 resembles the use of analogous quantities by Dobrushin[2] for higher dimensional models.

Definition 3.1. The configuration $\sigma=\{\sigma_i\}_{i\in\mathbf{Z}}$ with $\sigma_i=\pm 1$ is allowable if $x=\{x_i\}_{i\in\mathbf{Z}}$ is allowable, where

$$x_i=(-1)^i\sigma_i+1/2$$

A set of spin flips Γ defined by (2.4) is allowable if Γ corresponds to an allowable configuration σ.

As in (2.5), we make the identification $H_L(\sigma)=H_L(\Gamma)$ when σ corresponds to Γ via (2.4) and when σ and Γ are allowable.

Remark 3.3. It is easily checked that $\Gamma=\{(i_1, i_1+1),..., (i_{2n}, i_{2n}+1)\}$ is allowable if and only if the distance between any two consecutive spin flips in Γ is an odd integer and both $L-i_{2n}$ and i_1+L+1 are odd integers in the case that L is odd, and $L-i_{2n}$ and i_1+L+1 are even in the case that L is even.

Definition 3.2. Let $\gamma=\{(i_1, i_1+1),..., (i_{2n}, i_{2n}+1)\}\subset\Gamma$ be a primitive contour for the allowable contour Γ. Denote by $(\Gamma\backslash\gamma)'$ the collection of spin flips obtained from $\Gamma\backslash\gamma$ as follows: Translate all spin flips in $\Gamma\backslash\gamma$ which lie between i_k+1 and i_{k+1} one unit to the left if k is odd and $1\leqslant k<2n$. All other spin flips in $\Gamma\backslash\gamma$ remain unchanged.

It is easily checked that if Γ is allowable and $\gamma\subset\Gamma$ is primitive, then $(\Gamma\backslash\gamma)'$ is allowable.

Remark 3.4. The transformation which takes $\Gamma\backslash\gamma$ to $(\Gamma\backslash\gamma)'$ is related to the transformation $\tilde{T}_G$ of Dobrushin,[2] who considered Hamiltonians with hard core in $\mathbf{Z}^d$ for $d>1$. Our transformation is essentially a $|\gamma|/2$-fold composition of one-dimensional versions of Dobrushin's $\tilde{T}_G$.

Lemma 3.2. Let Γ be any allowable contour and let $\gamma \subset \Gamma$ be primitive. Then

$$|\tilde{H}(\Gamma \backslash \gamma) - \tilde{H}[(\Gamma \backslash \gamma)']| \leqslant \delta(M) \frac{\ln M}{M} L(\gamma) \tag{3.9}$$

where $\delta(M)$ is independent of Γ and γ, and $\lim_{M \to \infty} \delta(M) = 0$.

Proof. For an arbitrary collection of spin flips Γ_1, define for $i < j$

$$\chi_{\Gamma_1}(i, j) = \begin{cases} 1 & \text{if} \quad |\Gamma_1 \cap [i, j]| \text{ is odd} \\ 0 & \text{if} \quad |\Gamma_1 \cap [i, j]| \text{ is even} \end{cases}$$

Then

$$\begin{aligned} \tilde{H}[\Gamma \backslash \gamma] - \tilde{H}[(\Gamma \backslash \gamma)'] &= 2 \sum_{i<j} J_{ij}[\chi_{\Gamma \backslash \gamma}(i, j) - \chi_{(\Gamma \backslash \gamma)'}(i, j)] \\ &\equiv \sum_{i<j} U(\Gamma, \gamma, i, j) \end{aligned} \tag{3.10}$$

For the primitive contour γ, let I_k and $\bar{I}_k$ be as in Definition 2.1. From Definition 3.2

$$\chi_{\Gamma \backslash \gamma}(i, j) = \chi_{(\Gamma \backslash \gamma)'}(i, j) \tag{3.11}$$

when $i < j$, $i, j \in I_k$, and k is even. Thus,

$$\sum_{\substack{i<j \\ i, j \in I_k \backslash (\bar{I}_{k+1} \cup \bar{I}_{k-1})}} U(\Gamma, \gamma, i, j) = 0 \tag{3.12}$$

when k is even. Hence, we may write

$$\sum_{i<j} U(\Gamma, \gamma, i, j) = \sum_{k \text{ odd}} \sum_{\substack{i, j \in \bar{I}_k \\ i<j}} U(\Gamma, \gamma, i, j) + \sum{}^{*} U(\Gamma, \gamma, i, j) \tag{3.13}$$

where the sum $\sum^*$ in (3.13) is over all $i < j$ with not both i and j in $\bar{I}_k$ for k odd or in $I_k \backslash (\bar{I}_{k+1} \cup \bar{I}_{k-1})$ for k even. If $i+1$, $j+1 \in I_k$ or $i, j \in I_k$ for k odd, then

$$\chi_{(\Gamma \backslash \gamma)'}(i, j) = \chi_{\Gamma \backslash \gamma}(i+1, j+1) \tag{3.14}$$

and

$$J_{ij} = J_{i+1, j+1}$$

It follows that for k odd,

$$\left|\sum_{\substack{i,j\in \bar{I}_k \\ i<j}} U(\Gamma,\gamma,i,j)\right| \leqslant 2 \sum_{\substack{j\in \bar{I}_k \\ j\neq i_k}} J_{i_k j}\chi_{\Gamma\backslash\gamma}(i_k, j) + 2 \sum_{\substack{j\in \bar{I}_k \\ j\neq i_{k+1}+1}} J_{j i_{k+1}+1}\, \chi_{(\Gamma\backslash\gamma)'}(j, i_{k+1}+1) \quad (3.15)$$

If $j\notin I(\gamma_\alpha)$ for any primitive $\gamma_\alpha \subset I_k$, then

$$\chi_{\Gamma\backslash\gamma}(i_k, j) = 0 = \chi_{(\Gamma\backslash\gamma)'}(j-1, i_{k+1}+1)$$

Thus, the right side of (3.15) is bounded by

$$2\sum_{i<j} J_{ij}[\chi_{A_k}(i,j) + \chi_{A_k}(j,i)] \quad (3.16)$$

Applying Corollary 2.4 to (3.16) then gives

$$\left|\sum_{\substack{i,j\in \bar{I}_k \\ i<j}} U(\Gamma,\gamma,i,j)\right| < \frac{c_4}{\ln M}\frac{\ln M}{M}\ln(i_{k+1}-i_k) \quad (3.17)$$

Hence,

$$\left|\sum_{k\ \mathrm{odd}}\ \sum_{\substack{i,j\in \bar{I}_k \\ i<j}} U(\Gamma,\gamma,i,j)\right| < \frac{c_4}{\ln M}\frac{\ln M}{M} L(\gamma) \quad (3.18)$$

To bound the last sum on the right side of (3.13) we introduce the following terminology. Define $j\in \mathbf{Z}$ to be a "bad" point if there exists a primitive contour $\gamma_\alpha \subset I_k$ with k odd such that $(j, j+1) \subset \gamma_\alpha$. If a point in $\mathbf{Z}$ is not bad, we say it is "good."

It is easily checked that if i and j are both good or both bad, then

$$\chi_{\Gamma\backslash\gamma}(i,j) = \chi_{(\Gamma\backslash\gamma)'}(i,\ \ j)$$

Hence $\sum^* U(\Gamma, \gamma, i, j)$ is reduced to a sum over pairs i, j, where i and j are not both good or not both bad and not both in any $\bar{I}_k$. It follows that

$$\left|\sum{}^* U(\Gamma,\gamma,i,j)\right| \leqslant 2\sum_k \left[\sum_{i<j} J_{ij}(\chi_{A_k}(i,j) + \chi_{A_k}(j,i))\right] \quad (3.19)$$

Applying Corollary 2.4, we obtain

$$\left|\sum{}^* U(\Gamma,\gamma,i,j)\right| \leqslant \frac{c_4}{\ln M}\frac{\ln M}{M} L(\gamma) \quad (3.20)$$

Now combining (3.13), (3.18), and (3.20) gives (3.9) with $\delta(M)=c/\ln M$ and $c>0$.

Theorem 3.3. Let $\mu>\alpha$. For any allowable contour Γ and any primitive $\gamma\subset\Gamma$,

$$H(\Gamma)-H((\Gamma\backslash\gamma)')\geqslant L(\gamma)\left\{\frac{c_1}{(\ln M)^2}-[c_3+\delta(M)]\frac{\ln M}{M}\right\} \tag{3.21}$$

where c_1, $c_3>0$ and $\delta(M)$ is the same as in Lemma 3.2.

Proof. From Theorem 2.3,

$$\tilde{H}(\gamma)+\tilde{H}(\Gamma\backslash\gamma)-\tilde{H}(\Gamma)\leqslant c_3\frac{\ln M}{M}L(\gamma) \tag{3.22}$$

From Lemma 3.1 we also have

$$H(\Gamma)=\tilde{H}(\Gamma) \tag{3.23}$$

whenever Γ is allowable and when the constant c in (2.2) equals $\mu-\alpha$. Combining Lemma 3.2 with (3.22) and (3.23) gives

$$-H(\Gamma)-\delta(M)\frac{\ln M}{M}L(\gamma)+H((\Gamma\backslash\gamma)')+\tilde{H}(\gamma)\leqslant c_3\frac{\ln M}{M}L(\gamma)$$

or

$$H(\Gamma)-H((\Gamma\backslash\gamma)')\geqslant\tilde{H}(\gamma)-[c_3+\delta(M)]\frac{\ln M}{M}L(\gamma) \tag{3.24}$$

The proof is completed by combining (3.24) with Theorem 2.1.

As an immediate consequence of Theorem 3.3, we have the following.

Corollary 3.4. With the same hypotheses as in Theorem 3.3,

$$H(\Gamma)-H((\Gamma\backslash\gamma)')\geqslant\varepsilon L(\gamma) \tag{3.25}$$

where $\varepsilon>0$ for M sufficiently large.

Proof of Theorem 1.1. From Lemma 3.1 we may write

$$\frac{1}{2}\langle 1-\sigma_0\rangle_L^+=\frac{\sum_\Gamma e^{-\beta H(\Gamma)}\chi_0(\Gamma)}{\sum_\Gamma e^{-\beta H(\Gamma)}} \tag{3.26}$$

where the sums are over allowable contours Γ and where

$$\chi_0(\Gamma)=\begin{cases}1 & \text{if } \sigma_0(\Gamma)=-1\\ 0 & \text{if } \sigma_0(\Gamma)=+1\end{cases}$$

and $\sigma_0(\Gamma)$ is the spin value of σ_0 in the configuration according to Γ. Clearly $\chi_0(\Gamma)=0$ unless there is a primitive contour $\gamma \subset \Gamma$ with $I(\gamma) \ni 0$. Denote such a primitive contour by γ_Γ so that $\Gamma = \gamma_\Gamma \cup \gamma_2 \cup \cdots$ satisfies Condition D, with $0 \in I(\gamma_\Gamma)$. Then by Corollary 3.4 with M large enough so that $\varepsilon > 0$,

$$\begin{aligned}\frac{1}{2}\langle 1-\sigma_0\rangle_L^+ &\leqslant \left[\sum_\Gamma e^{-\beta\varepsilon L(\gamma_\Gamma)} e^{-\beta H((\Gamma\setminus\gamma_\Gamma)')}\chi_0(\Gamma)\right] \Big/ \sum_\Gamma e^{-\beta H(\Gamma)} \\ &\leqslant \sum_{\gamma:0\in I(\gamma)} e^{-\beta\varepsilon L(\gamma)} \sum_{\Gamma:\gamma=\gamma_\Gamma} e^{-\beta H((\Gamma\setminus\gamma)')} \Big/ \sum_\Gamma e^{-\beta H(\Gamma)} \\ &\leqslant \sum_{\gamma:I(\gamma)\in 0} e^{-\beta\varepsilon L(\gamma)} \end{aligned} \tag{3.27}$$

where all sums involving Γ are over allowable contours. From Theorem 2.2 it follows that

$$\frac{1}{2}\langle 1-\sigma_0\rangle_L^+ \leqslant \sum_{R\geqslant 1} e^{-\beta\varepsilon R} e^{c(R+1)} \tag{3.28}$$

and

$$\tfrac{1}{2}\langle 1-\sigma_0\rangle_L^+ < \tfrac{1}{2} \tag{3.29}$$

for β sufficiently large. Inequalities (3.28) and (3.29) hold uniformly in L. Since $\sigma_0 = 2x_0 - 1$, (3.29) implies (1.4). Repeating all of the above arguments with the boundary condition (1.2) replaced by (1.3) results in (1.5).

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